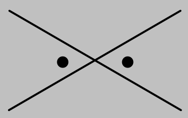
Geometry (Grades 11 & 12)

- Theorems
- Point of intersection theorems*
- Summary of reasons
- Method
- Proofs of theorems

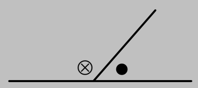
Geometry theorems: Grades 8 & 9

Lines of intersection

Vertically opposite ∠s



∠s on a straight line

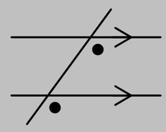


⊗ + • = 180°

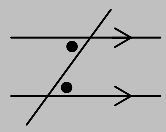
Revolution (∠s around point)

Parallel lines

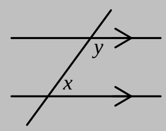
Corresponding ∠s



Alternate ∠s



Co-interior ∠s

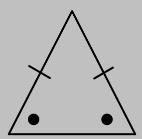


$x + y = 180°$

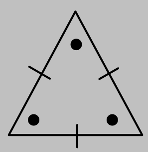
Triangles

Isosceles Δ

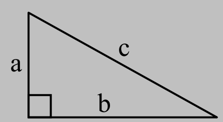
(∠s opposite equal sides)



Equilateral Δ: ∠s = 60°



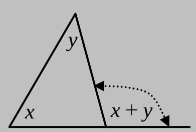
Right-angled Δ: (Pythagoras' theorem)



$a^2 + b^2 = c^2$

Sum of interior angles of Δ = 180°

Exterior ∠ of Δ = sum of interior opposite ∠s



Conditions for congruence (≡)

- side, side, side
- ∠, ∠, side
- side, included ∠, side
- 90°, hypotenuse, side

Area, circumference and volume

- Area rectangle = base × height
- Area Δ = ½ base × ⊥height
- Area O = $\pi . r^2$
- Area trapezium
  = ½ (sum of // sides) × height
- Circumference O = $2\pi . r$
- Volume (cylinder / prism)
  = base area × height

Geometry theorems: Grade 10

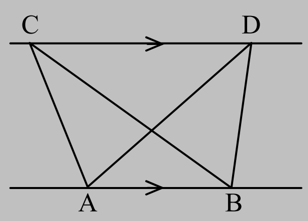
Quadrilaterals

- The sum of the interior ∠s of quadrilateral = 360°

Properties	Rectangle	Rhombus	Parm	Square	Kite	Trapezium	Quadrilateral
One pair of // sides	✓	✓	✓	✓		✓	
Two pairs of // sides	✓	✓	✓	✓			
∠s = 90°	✓			✓			
All sides equal		✓		✓			
Two pairs of adjacent sides equal		✓		✓	✓		
Diagonals equal	✓			✓			
Diagonals bisect each other	✓	✓	✓	✓			
Diagonals bisect each other rectangularly		✓		✓	✓		

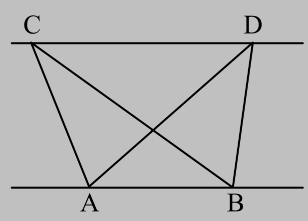
AB ∥ CD ⇒ Area ΔABC = Area ΔABD

(same base & same // lines)



Area ΔABC = Area ΔABD ⇒ AB ∥ CD

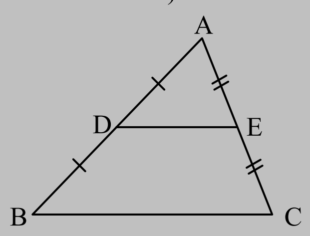
(same base & areas equal)



AD = DB & AE = EC

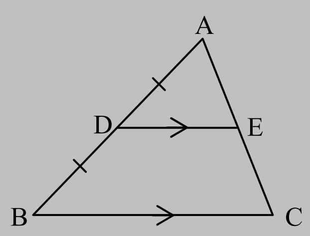
⇒ BC ∥ DE & DE = ½ BC

(centre theorems)



BC ∥ DE & AD = DB ⇒ AE = EC

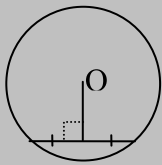
(line from centre & // lines)



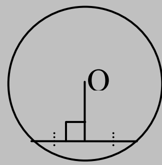
Geometry Theorems: Grade 11

Geometry I: Angles & chords

Theorem 1(a) HG/SG
Line through centres of O and chord

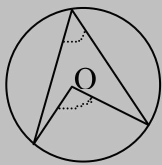


Theorem 1(b) HG/SG
Line from centre ⊥ chord

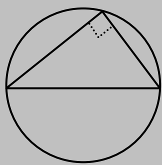


Theorem 2 HG/SG

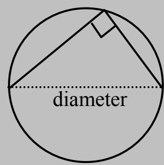
∠ at centre = 2 × ∠ at circumference



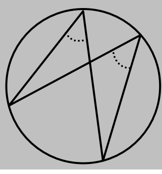
Theorem 3(a)
∠ in semi O



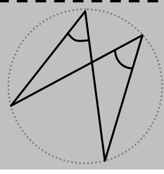
Theorem 3(b)
Chord spans 90°∠



Theorem 4(a)
∠s at circumference in the same O segment



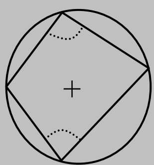
Theorem 4(b)
AD spans equal ∠s



Geometry II: Cyclic quadrilateral

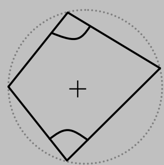
Theorem 5(a) HG/SG

Opposite ∠s of cyclic quadrilateral



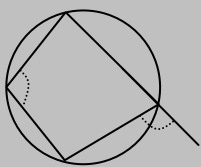
= 180°

Theorem 5(b) HG/SG
Opposite ∠s supplementary

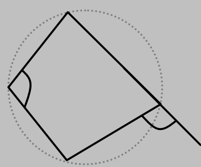


= 180°

Theorem 6(a)
Exterior ∠ = interior opposite ∠



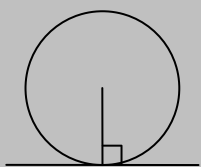
Theorem 6(b)
Exterior ∠ = interior opposite ∠



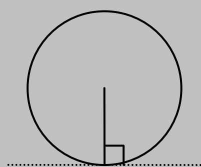
Prove a cyclic quadrilateral

Geometry III: Tangents to circles

Theorem 7(a)
Tangent ⊥ radius

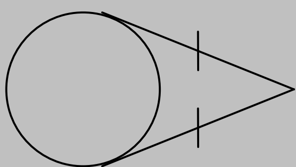


Theorem 7(b)
Line ⊥ radius

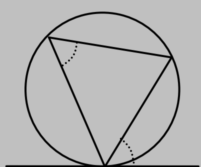


Theorem 8

Tangents from the same point

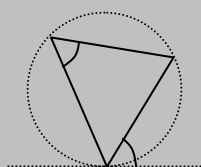


Theorem 9(a) HG/SG
∠ between tangent & chord

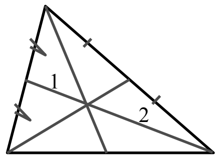
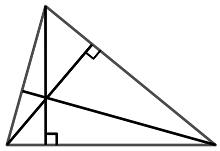
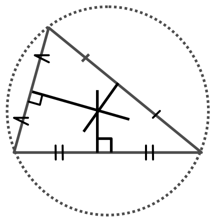
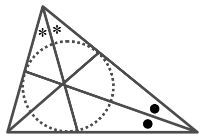


Theorem 9(b) HG/SG

∠ between line & chord



Point of intersection theorems *

	Definition	Point of intersection	Sketch	Property
Median	A median is a line drawn from the vertex of a triangle to the centre of the opposite side.	Median point		Third splitting point of median
Altitude	An altitude of a triangle is a line drawn from the vertex of the triangle perpendicular to the opposite side.	Orthocentre		Divides Δ in 3 cyclic quadrilaterals
Perpendicular bisector	A perpendicular bisector of a line segment AB is a perpendicular line dividing the line segment AB.	Centre of circumcircle		Centre of circumcircle
Bisector of an angle	A bisector of an angle is a line halving the angle.	Centre of incircle		Centre of incircle

Theorem 10 *: The medians of a triangle intersect at one point, namely the median point.
(Medians of Δ)

Theorem 11 *: The bisectors of the angles intersect at one point, namely the centre of the incircle.
(Bisectors of ∠s of Δ)

Theorem 12 *: The perpendicular bisectors of a triangle intersect at one point, namely the centre of the circumcircle.
(Perpendicular bisectors of sides of Δ)

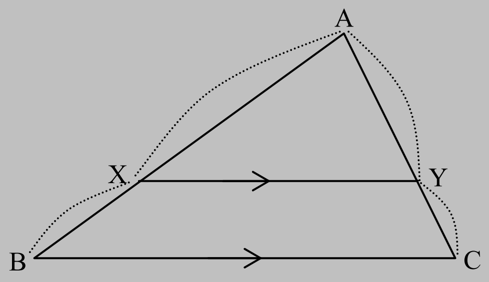
Theorem 13 *: Altitudes of a triangle intersect at one point, namely the orthocentre.
(Altitudes of Δ)

Geometry theorems: Grade 12

Proportionality & parallelism

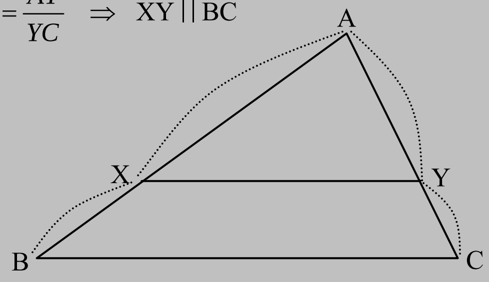
Theorem 1a (HG & SG): (Line || to side of Δ)

$XY \parallel BC \Rightarrow \frac{AX}{XB} = \frac{AY}{YC}$



Theorem 1b (HG): (Line divides sides of Δ proportionally)

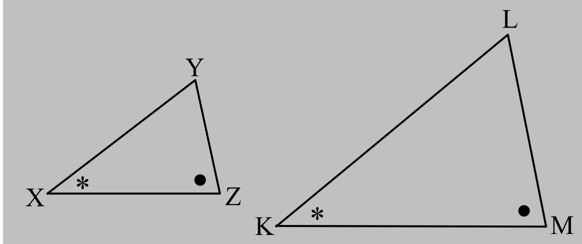
$\frac{AX}{XB} = \frac{AY}{YC} \Rightarrow XY \parallel BC$



Proportionality & similarity

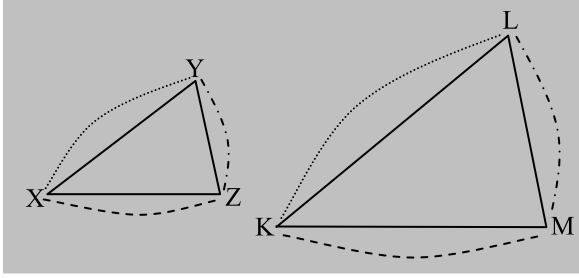
Theorem 2a (HG & SG): (Equiangular Δs)

$\hat{X} = \hat{K}$, $\hat{Y} = \hat{L}$, $\hat{Z} = \hat{M} \Rightarrow \frac{KL}{XY} = \frac{LM}{YZ} = \frac{KM}{XZ}$



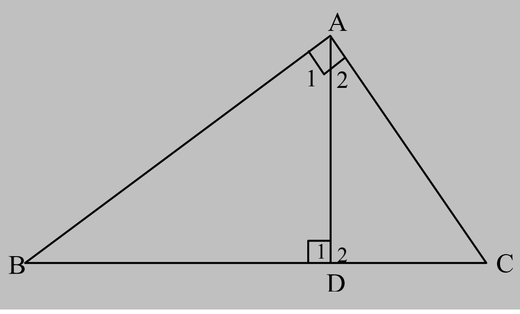
Theorem 2b (HG): (Sides of Δs proportional)

$\frac{KL}{XY} = \frac{LM}{YZ} = \frac{KM}{XZ} \Rightarrow \hat{X} = \hat{K}$, $\hat{Y} = \hat{L}$, $\hat{Z} = \hat{M}$



*** Theorem 3 (HG):** (Uniform right-angled Δs)

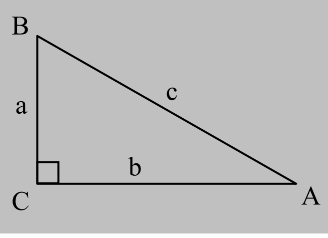
$AD \perp BC$ & $\hat{A} = 90° \Rightarrow \Delta ABD \,|||\, \Delta CAD \,|||\, \Delta CBA$



Pythagoras' theorem

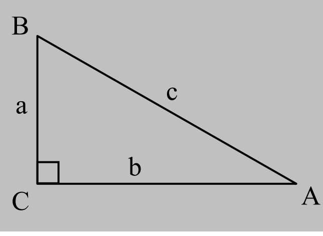
*** Theorem 4a:** (Pythagoras' theorem)

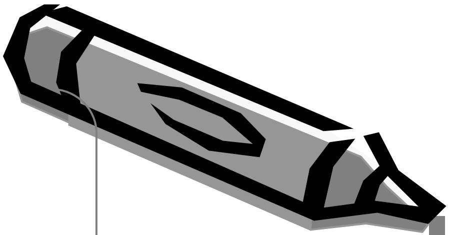
$\hat{C} = 90° \Rightarrow a^2 + b^2 = c^2$



*** Theorem 4b:** (Converse of Pythagoras)

$a^2 + b^2 = c^2 \Rightarrow \hat{C} = 90°$





Reasons for geometry theorems

Lines of intersection	Vertically opposite ∠s
	∠s on a straight line
	Revolution (∠s around point)
Triangles	∠s opposite equal sides
	Equilateral Δ: ∠s = 60°
	Pythagoras' theorem
	Sum of interior angles of Δ
	Exterior ∠ of Δ
Parallelism	Corresponding ∠s
	Alternate ∠s
	Co-interior ∠s
Congruence	side, side, side
	∠, ∠, side
	side, included ∠, side
	90°, hypotenuse, side

Gr 8 & 9

Area of Δ	Same base & same // lines
	Same base & areas equal
	Centre theorem
	Line from centre & // lines

Gr 10

Proofs of Theorems ⇒

Angles & chords

Theorem 1a:	Line through centres of O and chord	**HG & SG**
Theorem 1b:	Line from centre ⊥ chord	**HG**
Theorem 2:	∠ at centre = 2 × ∠ at circumference	**HG & SG**
Theorem 3a:	∠ in semi O	
Theorem 3b:	Chord spans 90°∠	
Theorem 4a:	∠s at circumference in same O segment	
Theorem 4b:	AD spans equal ∠s	

Cyclic quadrilaterals

Theorem 5a:	Opposite ∠s of cyclic quadrilateral	**HG & SG**
Theorem 5b:	Opposite ∠s supplementary	**HG**
Theorem 6a:	Exterior ∠ = interior opposite ∠	
Theorem 6b:	Exterior ∠ = interior opposite ∠	

Tangents to circles

Theorem 7a:	Tangent ⊥ radius	
Theorem 7b:	Line ⊥ radius	
Theorem 8:	Tangents from the same point	
Theorem 9a:	∠ between tangent & chord	**HG & SG**
Theorem 9b:	∠ between line & chord	**HG**

Point of intersection theorems (HG only)

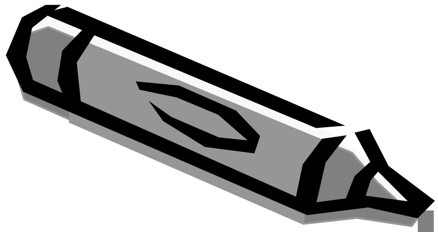
Theorem 10:	Medians of Δ intersect at 1 point
Theorem 11:	Bisectors of ∠s of Δ intersect at 1 point
Theorem 12:	Perpendicular bisectors of sides of Δ intersect at 1 point
Theorem 13:	Altitudes of Δ intersect at 1 point

Gr 11

Similarity and proportionality

Theorem 1a:	Line || to side of Δ	**HG & SG**
Theorem 1b:	Line divides 2 sides of Δ proportionally	**HG**
Theorem 2a:	Equiangular Δs	**HG & SG**
Theorem 2b:	Sides of Δs proportional	**HG**
Theorem 3:	Uniform right-angled Δs	**HG**
Theorem 4a:	Pythagoras	

Gr 12



Method to solve problems

Step 1: Mark all given information on the sketch.

- Make parallel lines and tangent lines different colours.

Step 2: Expand on the given information.

- Centre of a circle
  - ⇒ line from centre on chord
  - ⇒ radius ⊥ tangent
  - ⇒ isosceles Δ with radii
  - ⇒ ∠ at centre = 2 × ∠ at circumference
  - ⇒ ∠ in semi ○
- Parallel lines
  - ⇒ alternate ∠s
  - ⇒ corresponding ∠s
  - ⇒ interior ∠s
  - ⇒ sides of Δs proportional (Gr 12)
- Cyclic quadrilateral
  - ⇒ opposite ∠s of quadrilateral supplementary
  - ⇒ exterior ∠ of cyclic quadrilateral
  - ⇒ ∠s at circumference in same ○ segment
- Tangents
  - ⇒ tangent ⊥ radius
  - ⇒ ∠ between tangent and chord
  - ⇒ two tangents from point

Step 3: Examine question: Write in abstract form.
Abstract form means to express what is required in terms of angles in the sketch.

If the following is required:

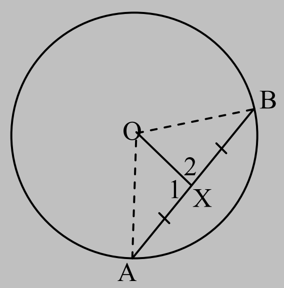
- Prove that two lines are parallel, then prove:
  - ⇒ alternate ∠s are equal
  - ⇒ corresponding ∠s are equal
  - ⇒ interior ∠s on the same side are supplementary
- Prove that a triangle is isosceles, then prove:
  - ⇒ the opposite angles are equal
- Prove that a quadrilateral is a cyclic quadrilateral (concyclic), then prove:
  - ⇒ opposite angles of quadrilateral are supplementary
  - ⇒ exterior ∠ of quadrilateral is equal to interior opposite ∠
  - ⇒ line segment spans equal ∠s
- Prove that a line is a tangent, then prove:
  - ⇒ line ⊥ radius
  - ⇒ ∠ between line and chord is equal to opposite ∠

Step 4: Perform proof.

Proofs: Grade 11 geometry theorems

Theorem 1a (HG & SG): The join of the centre of a circle and the centre of a chord is normal to the chord.

Given: AX = XB
To prove: OX $\perp$ AB
Construction: Join OA and OB.

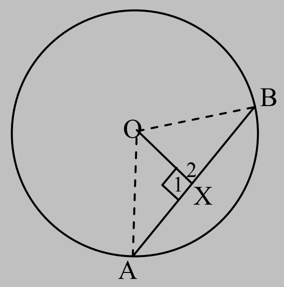


In ΔAOX and ΔBOX:

OX = OX	(common)
OA = OB	(radii)
AX = XB	(given)
$\therefore$ ΔAOX $\equiv$ ΔBOX	(side, side, side)
$\therefore \hat{X}_1 = \hat{X}_2$	(congruence)
but $\hat{X}_1 + \hat{X}_2 = 180°$	($\angle$s on straight line)
$\therefore \hat{X}_1 = \hat{X}_2 = 90°$	
$\therefore$ OX $\perp$ AB	

Theorem 1b (HG): The normal from the centre of a circle to any chord bisects the chord.

Given: OX $\perp$ AB
To prove: AX = XB
Construction: Join OA and OB.

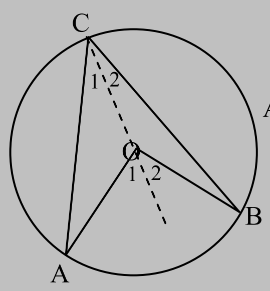


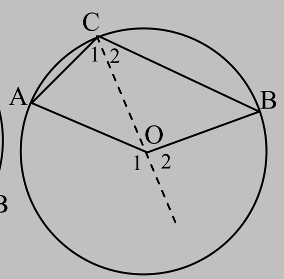
In ΔAOX and ΔBOX:

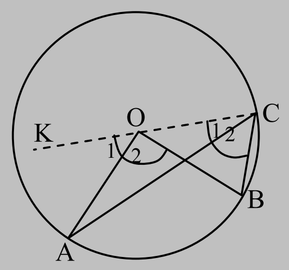
OX = OX	(common)
OA = OB	(radii)
$\hat{X}_1 = \hat{X}_2 = 90°$	(given)
$\therefore$ ΔAOX $\equiv$ ΔBOX	(90°, side, side)
$\therefore$ AX = XB	(congruence)

Theorem 2 (HG & SG): The angle spanning an arc of a circle at the centre is double the angle that it spans at any point on the circumference.

To prove: $A\hat{O}B = 2A\hat{C}B$
Construction: Join O with C.







$\hat{O}_2 = K\hat{O}B$
& $\hat{C}_2 = K\hat{C}B$

<u>Sketches 1 & 2</u>:

$\hat{O}_1 = \hat{A} + \hat{C}_1$	(exterior $\angle$ of Δ)
$\hat{O}_1 = 2\hat{C}_1$	(OA = OC; radii)

Similarly $\hat{O}_2 = 2\hat{C}_2$

$\therefore \hat{O}_1 + \hat{O}_2 = 2(\hat{C}_1 + \hat{C}_2)$

$\therefore A\hat{O}B = 2A\hat{C}B$

<u>Sketch 3</u>:

$\hat{O}_1 = \hat{A} + \hat{C}_1$	(exterior $\angle$ of Δ)
$\hat{O}_1 = 2\hat{C}_1$	(OA = OC; radii)

Similarly $\hat{O}_2 = 2\hat{C}_2$

$\therefore \hat{O}_2 - \hat{O}_1 = 2(\hat{C}_2 - \hat{C}_1)$

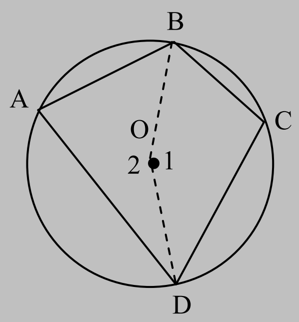
$\therefore A\hat{O}B = 2A\hat{C}B$

Theorem 5a (HG & SG): The opposite angles of a cyclic quadrilateral are supplementary.

Given: ABCD a cyclic quadrilateral

To prove: $\hat{A}+\hat{C}=180°$, $\hat{B}+\hat{D}=180°$

Construction: Join OB and OD.



$\hat{O}_1=2\hat{A}$ & $\hat{O}_2=2\hat{C}$ ($\angle$ at centre = 2 × $\angle$ at circumference)

$\therefore \hat{O}_1+\hat{O}_2=2(\hat{A}+\hat{C})$

but $\hat{O}_1+\hat{O}_2=360°$ (revolution)

$\therefore 2(\hat{A}+\hat{C})=360°$

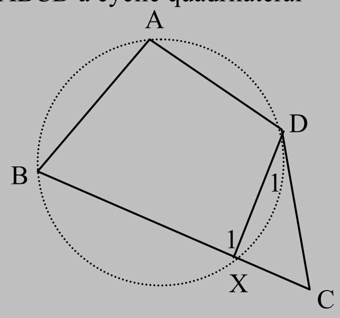
$\therefore \hat{A}+\hat{C}=180°$

$\therefore \hat{B}+\hat{D}=180°$ (sum of interior $\angle$s of quadrilateral)

Theorem 5b (HG): If a pair of opposite angles of a quadrilateral is supplementary, then the quadrilateral is a cyclic quadrilateral.

Given: $\hat{A}+\hat{C}=180°$

To prove: ABCD a cyclic quadrilateral



Assume that C does not lie on the circle.

Now draw a circle through A, B and D. Join DX.

Therefore $\hat{A}+\hat{C}=180°$ (given)

and $\hat{A}+\hat{X}_1=180°$ (opp. $\angle$s of cyclic quad ABDX)

$\therefore \hat{C}=\hat{X}_1$

This is impossible because $\hat{X}_1=\hat{D}_1+\hat{C}$ (ext. $\angle$ of Δ)

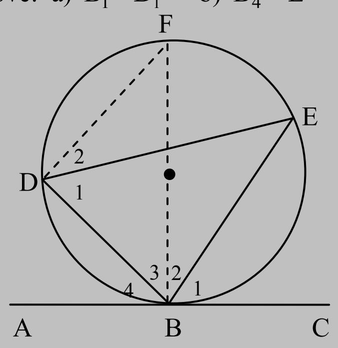
$\therefore$ Assumption was wrong.

$\therefore$ C lies on circle, therefore ABCD a cyclic quadrilateral.

Theorem 9a (HG & SG): The angle formed by a tangent to a circle and a chord drawn from the tangent is equal to an angle in the opposite circle segment.

Given: AC is a tangent.

To prove: a) $\hat{B}_1=\hat{D}_1$ b) $\hat{B}_4=\hat{E}$



a) Draw diameter BF and join FD.

$\hat{B}_1+\hat{B}_2=90°$ (radius $\perp$ tangent)

$\hat{D}_1+\hat{D}_2=90°$ ($\angle$ in semi O)

but $\hat{B}_2=\hat{D}_2$ ($\angle$s at circumference on FE)

$\therefore \hat{B}_1=\hat{D}_1$

b) $\hat{D}_1+(\hat{B}_2+\hat{B}_3)+\hat{E}=180°$ (interior $\angle$s of Δ)

$\hat{B}_1+(\hat{B}_2+\hat{B}_3)+\hat{B}_4=180°$ ($\angle$s on straight line)

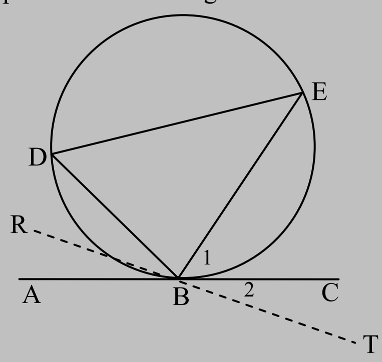
but $\hat{B}_1=\hat{D}_1$ (already proved)

$\therefore \hat{B}_4=\hat{E}$

Theorem 9b (HG): If a line through the terminal of a chord forms an angle with the chord equal to an angle in the opposite segment, then the line is a tangent to the circle.

Given: $\hat{B}_1=\hat{D}$

To prove: AC is a tangent.



Assume that AC is not a tangent, but RT is one.

$\hat{B}_1+\hat{B}_2=\hat{D}$ ($\angle$ between tangent & chord)

but $\hat{B}_1=\hat{D}$ (given)

$\therefore \hat{B}_1+\hat{B}_2=\hat{B}_1$ (both equal to $\hat{D}$)

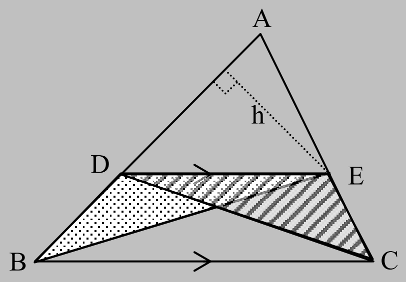
This is false.

$\therefore$ Assumption that RT is a tangent is false.

$\therefore$ AC is a tangent.

Proofs: Grade 12 geometry theorems

Theorem 1a (HG & SG): A line parallel to one side of a triangle divides the other two in proportional sections.



Given: DE││BC

To prove: $\frac{AD}{BD} = \frac{AE}{EC}$

Construction: Draw DC and BE.

$\frac{Area\Delta ADE}{Area\Delta BDE} = \frac{\frac{1}{2}AD.h}{\frac{1}{2}BD.h} = \frac{AD}{BD}$ (same height h)

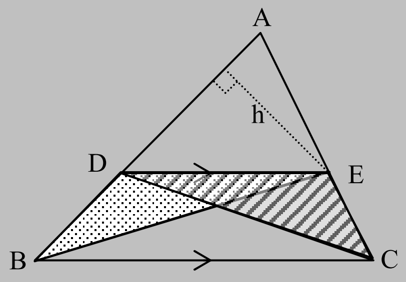
Similarly $\frac{Area\Delta ADE}{Area\Delta DEC} = \frac{AE}{EC}$

But area ΔBDE = area ΔDEC (Δs on the same base and parallel lines)

$\therefore \frac{Area\Delta ADE}{Area\Delta BDE} = \frac{Area\Delta ADE}{Area\Delta DEC}$

$\therefore \frac{AD}{BD} = \frac{AE}{EC}$

Theorem 1b (HG): If a line divides two sides of a triangle proportionally, then the line is parallel to the third side.



Given: $\frac{AD}{BD} = \frac{AE}{EC}$

To prove: DE││BC

Construction: Draw DC and BE.

$\frac{Area\Delta ADE}{Area\Delta BDE} = \frac{\frac{1}{2}AD.h}{\frac{1}{2}BD.h} = \frac{AD}{BD}$ (same height h)

Similarly $\frac{Area\Delta ADE}{Area\Delta DEC} = \frac{AE}{EC}$

But $\frac{AD}{BD} = \frac{AE}{EC}$ (given)

Therefore $\frac{Area\Delta ADE}{Area\Delta BDE} = \frac{Area\Delta ADE}{Area\Delta DEC}$

∴ Area ΔBDE = Area ΔDEC

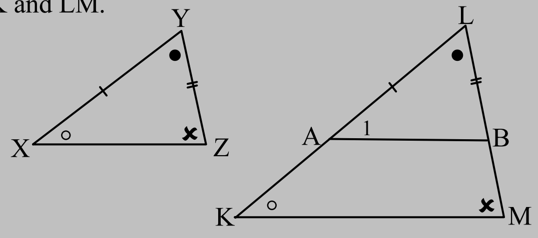
∴ DE││BC (same base DE)

Theorem 2a (HG & SG): If two triangles are equiangular, then the lengths of their corresponding sides are proportional.

Given: ΔXYZ & ΔKLM with $\hat{X} = \hat{K}$, $\hat{Y} = \hat{L}$, $\hat{Z} = \hat{M}$

To prove: $\frac{KL}{XY} = \frac{LM}{YZ} = \frac{KM}{XZ}$

Construction: Mark off LA = XY and LB = YZ on LK and LM.



$\Delta LAB \equiv \Delta YXZ$ $(s, \angle, s)$

$\therefore \hat{A}_1 = \hat{X}$

But $\hat{X} = \hat{K}$ (given)

$\therefore \hat{A}_1 = \hat{K}$

$\therefore AB \,||\, KM$ (corresponding $\angle$s)

$\therefore \frac{KL}{AL} = \frac{LM}{LB}$ $(AB \,||\, KM)$

But AL = XY and LB = YZ

$\therefore \frac{KL}{XY} = \frac{LM}{YZ}$

Similarly, by marking off XY and XZ on KL and KM, it can be shown that $\frac{KL}{XY} = \frac{KM}{XZ}$

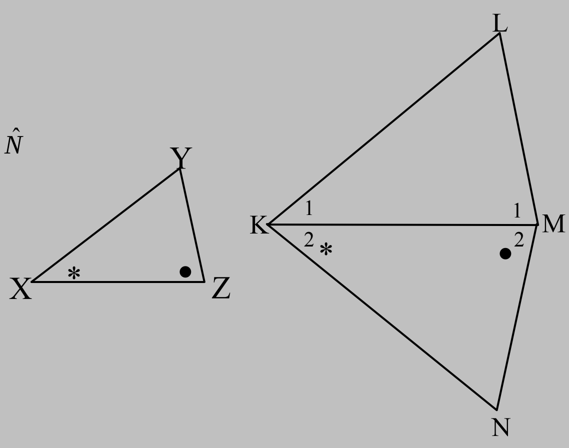
$\therefore \frac{KL}{XY} = \frac{LM}{YZ} = \frac{KM}{XZ}$

Theorem 2b (HG): If the lengths of the corresponding sides of two triangles are proportional, then the two triangles are equiangular.

Given: ΔXYZ & ΔKLM with $\frac{KL}{XY} = \frac{LM}{YZ} = \frac{KM}{XZ}$

To prove: $\hat{X} = \hat{K}_1$, $\hat{Y} = \hat{L}$, $\hat{Z} = \hat{M}_1$

Draw ΔKMN on KM so that $\hat{X} = \hat{K}_2$, $\hat{Z} = \hat{M}_2$, $\hat{Y} = \hat{N}$



$\therefore \Delta XYZ$ and ΔKNM are equiangular.

$\therefore \frac{KN}{XY} = \frac{KM}{XZ}$ (equiangular Δs)

but $\frac{KL}{XY} = \frac{KM}{XZ}$ (given)

$\therefore$ KL = KN

Similarly ML = MN

Therefore $\Delta KLM \equiv \Delta KNM$ (side, side, side)

$\therefore \hat{K}_2 = \hat{K}_1$ (congruence)

but $\hat{X} = \hat{K}_2$ (given)

$\therefore \hat{X} = \hat{K}_1$

Similarly $\hat{Z} = \hat{M}_1$ and $\hat{Y} = \hat{L}$

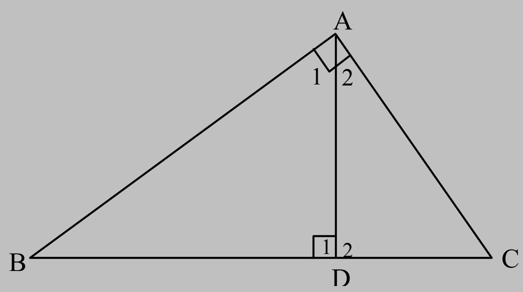
$\therefore \Delta XYZ \,///\, \Delta KLM$ $(\angle, \angle, \angle)$

Therefore $\hat{X} = \hat{K}_1$, $\hat{Y} = \hat{L}$, $\hat{Z} = \hat{M}_1$

Theorem 3 (HG): The normal from the right angle vertex of a right-angled triangle on the hypotenuse divides the triangle in two triangles that are uniform to each other and to the original triangle.

Given: ΔABC with $\hat{A}_2 + \hat{A}_1 = 90°$ and $\hat{D}_1 = 90°$

To prove: ΔABD ||| ΔCAD ||| ΔCBA



$\hat{B} + \hat{A}_1 = 90°$ ($\hat{D}_1 = 90°$)

$\hat{A}_2 + \hat{A}_1 = 90°$ (given)

$\therefore \hat{B} = \hat{A}_2$ Similarly $\hat{A}_1 = \hat{C}$

In ΔABD, ΔCAD and ΔCBA:

$\hat{B} = \hat{A}_2 = \hat{B}$ (proved)

$\hat{A}_1 = \hat{C} = \hat{C}$ (proved)

$\hat{D}_1 = \hat{D}_2 = B\hat{A}C$ (proved)

$\therefore$ ΔABD ||| ΔCAD ||| ΔCBA (uniform Δs)

Deductions (NB: Memorise)

- ΔCBA ||| ΔABD $\Rightarrow$ $AB^2 = BD.BC$
- ΔABD ||| ΔCAD $\Rightarrow$ $AD^2 = BD.DC$
- ΔCBA ||| ΔCAD $\Rightarrow$ $AC^2 = DC.BC$